時變數據的尖銳型高低通濾波器

Approximate Sharp High/Low-Passed Filters

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ABSTRACT

Two sharp high/low passed filters are developed. The first filter bases on the iterative and diffusive filter and a simple Fast Fourier Transform (FFT) algorithm and the second filter employs the linear trend removal and the FFT algorithm. The iterative filter is modified to ensure the non-negative property. The first procedure of the proposed sharp filter involves the following steps: uses the iterative filter to define the cutting point on the frequency domain; evaluates the spectrum of the high frequency part generated by the iterative filter; removes all modes with frequencies smaller than the cut-off frequency; performs the inverse FFT to obtain the high frequency part; and the difference between the original data string and high frequency part is the non-sinusoidal and low frequency part. Band-passed limited spectrums are then directly evaluated by imposing desired band on the spectrum. The second filter following the same procedure except that the smooth part is summing up the linear removed part and the smoothed part evaluated by the low frequency modes with frequency smaller than the cut-off frequency. A test case of low speed turbulent flow data is employed to show the performances of these two filters. From the resulting data strings corresponding to several bandwidths, which have sharp cut-off resolution around cut-off frequencies, show that these two filters have similar performance and are very fast.

Keywords: sharp filters, iterative and diffusive filter, spectrum with small error.

Introduction

Because of the rapid development of computer hardware and software, the capability of collecting huge number of long data string increases rapidly. Generally, a real data string frequently has a complex structure which changes main characters rapidly. The current method considers the Fourier spectrum as a parameter representation of time series data, $y_0, y_1, ..., y_n$, where n is the data size. In fact, the Fourier spectrum is an exact parameter representation of the original data only if the data and all the derivatives up to $(n-1)$ th order are periodic. Unfortunately, most data string can not satisfy these restrictions simultaneously so that the corresponding discrete Fourier spectrum represents the original data string in a weak sense. According to the authors' experience, most of these Fourier spectrums frequently involve certain minor errors which reflect a fraction of information of the dominant modes and contaminate almost all minor modes. In Ref.[1-7], an approximate Fourier sine spectrum was proposed to replace the Fourier spectrum. Note that a Fourier sine spectrum requires $y = 0$ at two ends and odd function properties. In order to satisfy these requirements, finite segments around the two ends should be discarded and the non-sinusoidal part must be removed. Therefore, most of the resulting Fourier sine spectrums may slightly deviate form their corresponding Fourier spectrums.

In Ref.[8,9], The non-sinusoidal part is estimated by the smooth part generated by the iterative and diffusive filter. Unfortunately, the transition zone of the filter cannot be made sharp enough with a reasonable computing time. This study will propose a modified version of the iterative filter and an simple filter without the iterative procedure.

Analysis

The Iterative Filter Basing on Gaussian Smoothing

Assume that a discrete data string can be approximated by

$$y(t_i)=\sum_{n=0}^{N} b_n \cos\left(\frac{2\pi t_i}{\lambda_n}\right)+c_n \sin\left(\frac{2\pi t_i}{\lambda_n}\right) \quad (1)$$

In Ref.[5,6], it was proven that after applying the Gaussian smoothing once, the resulting smoothed data becomes

$$\bar{y}(t_i)\approx\sum_{n=0}^{N} a(\sigma/\lambda_n)\{b_n \cos\left(\frac{2\pi t_i}{\lambda_n}\right)+c_n \sin\left(\frac{2\pi t_i}{\lambda_n}\right)\} \quad (2)$$

where $a(\sigma/\lambda_n)$ is the attenuation factor introduced by the smoothing and takes the following approximate form in the interior points remote from the two ends (say $|i-i_{boundary}|\Delta t>4\sigma$).

$$a(\sigma,\lambda_l)\approx\frac{1}{k}\sum_{i=-\infty}^{\infty} e^{-t_i^2/(2\sigma^2)}\cos\frac{2\pi t_i}{\lambda_l}$$
$$k\approx\sum_{i=-\infty}^{\infty} e^{-t_i^2/(2\sigma^2)} \quad (3)$$

It can also be proven that

$$TV(\bar{y})=\sum_{i=-\infty}^{\infty}|\bar{y}_i-\bar{y}_{i-1}|\le\sum_{i=-\infty}^{\infty}|y_i-y_{i-1}|=TV(y) \quad (4)$$

From Eqs.(3) and (4), it can be easily proven that

$$a(\sigma/\lambda_n)\approx\exp[-2\pi^2\sigma^2/\lambda_n^2]\le 1 \quad (5)$$

The negative property of $a(\sigma/\lambda_n)$ can only be proven numerically and may be violated by the machine error of the computing device. If the Gaussian smoothing is employed to smoothed $\bar{y}_i=\bar{y}(t_i)$ again to obtain

$$\bar{y}_{1,i}=\bar{y}_1(t_i)\approx$$
$$\sum_{n=0}^{N} a^2(\sigma/\lambda_n)\{b_n \cos\left(\frac{2\pi t_i}{\lambda_n}\right)+c_n \sin\left(\frac{2\pi t_i}{\lambda_n}\right)\} \quad (6)$$

The attenuation factor will satisfy the diffusive property, say

$$0\le a^2(\sigma/\lambda_n)\approx\exp[-4\pi^2\sigma^2/\lambda_n^2]\le 1 \quad (7)$$

If the removed high frequency part is denoted as $y'_{i,1}$ and apply the same smoothing twice to it to obtain the second smoothed result as $\bar{y}_{i,2}$ and repeat the same procedure to obtain the $m-$th smoothed and high frequency part as $\bar{y}_{m,i}$ and $y'_{m,i}$, respectively. The following relation can be built

$$y'_{m,i}=\sum_{n=0}^{N}[1-a^2(\sigma/\lambda_n)]^m\left[b_n \cos\left(\frac{2\pi t_i}{\lambda_n}\right)+c_n \sin\left(\frac{2\pi t_i}{\lambda_n}\right)\right]$$

$$\bar{y}_i(m)=\bar{y}_{1,i}+\bar{y}_{2,i}+\ldots+\bar{y}_{m,i}=y_i-y'_{m,i}$$
$$=\sum_{n=0}^{N} A_{n,m,\sigma}\left[b_n \cos\left(\frac{2\pi t_i}{\lambda_n}\right)+c_n \sin\left(\frac{2\pi t_i}{\lambda_n}\right)\right] \quad (8)$$

$\bar{y}_i(m)$ can be considered as the smoothed part and $y'_{m,i}$ as the high frequency part. After substituting Eq.(7), in the interior points remote from the two ends, the attenuation factor of employing the Gaussian smoothing method satisfies

$$A_{n,m,\sigma}\approx 1-[1-\exp\{-4\pi^2\sigma^2/\lambda_n^2)]^m$$
$$0\le A_{n,m,\sigma}\le 1 \quad (9)$$

Obviously, the accumulated smooth part is embedded by a diffusive attenuation factor with negligible phase error in the interior point remote from the two ends.

Suppose that one want to cut the spectrum at λ_c, all modes with $\lambda<\lambda_c$ will be preserved while the rest modes with $\lambda\ge\lambda_c$ together with the non-sinusoidal part will removed. It should be noted that a discrete Fourier expansion will embed spectrum components almost running over the whole spectrum to reflect the non-sinusoidal part. These Fourier components involve two parts: the original non-sinusoidal part and embedded part to enforce the two ends of non-sinusoidal part to be periodic. The embedded part has a significant magnitude around the two ends and rapidly attenuates to an insignificant magnitude in the interior points remote from the two ends. If the interest region is restricted in the interior part where $4\sigma<t_i<t_n-4\sigma$, all the embedded Fourier components have insignificant contribution [10]. In other words, the modes with $\lambda\ge\lambda_c$ can properly represent the non-sinusoidal and low frequency parts in the interior points. Based on this fact, the following equations are employed to solve the desired parameters σ and number of iteration steps m.

$$1-[1-\exp(-4\pi^2\sigma^2/\lambda_c^2)]^m=0.999$$
$$1-[1-\exp(-4\pi^2\sigma^2/\bar{\lambda}^2)]^m=0.001 \quad (10)$$

where $\bar{\lambda}\le 0.1t_n$ is employed to ensure a sufficient wide range with an accurate of Fourier resolution.

Before the FFT algorithm of Ref.[3] is employed. One can use a classical FFT algorithm to obtain a spectrum. After inspecting both the original data and spectrum, he can decide a suitable transition range defined by λ_c and $\bar{\lambda}$ to remove the estimated non-sinusoidal and low frequency part.

Monotonic Cubic Interpolation Method

For the sake of simplicity and to employ the data structure of a computer, it seems convenient to use the simple FFT algorithm whose data points are exactly equal to 2^m. Because a real data string may not have exactly the number of integer powers of 2 in uniform spacing form, an interpolation scheme should be employed to redistribute the points. In Ref.[1], a Hermite cubic interpolation between points (t_i, t_{i+1}) is employed, say,

$$y(t) = c_3(t-t_i)^3 + c_2(t-t_i)^2 + c_1(t-t_i) + c_0$$
$$c_0 = y_i,\ c_1 = y_i',\ s_{i+1/2} = (y_{i+1}-y_i)/(t_{i+1}-t_i) \quad (11)$$

$$c_2 = \frac{3s_{i+1/2} - 2y_i' - y_{i+1}'}{t_{i+1}-t_i},\ c_3 = \frac{y_i' + y_{i+1}' - 2s_{i+1/2}}{(t_{i+1}-t_i)^2}$$

In order to avoid spurious oscillation, Ref.[1] modified the M3A interpolation of Ref.[11] to give necessary limiter to slopes y_i', y_{i+1}' as

$$y_i' = \mathrm{sgn}(k_i)$$
$$\min\left[\frac{1}{2}\left|p_{i-1/2}'(t_i) + p_{i+1/2}'(t_i)\right|, \max\left\{\bar{k}\,|s_i|, \frac{\bar{k}}{2}|k_i|\right\}\right]$$

$$\begin{aligned}
&p_{i-1/2}'(t_i) = s_{i-1/2} + d_{i-1/2}(t_i - t_{i-1}),\\
&p_{i+1/2}'(t_i) = s_{i+1/2} + d_{i+1/2}(t_i - t_{i+1}),\\
&k_i = \min\mathrm{mod}\left[p_{i-1/2}'(t_i), p_{i+1/2}^{i}(t_i)\right],\\
&d_{i+1/2} = \min\mathrm{mod}\left[d_i, d_{i+1}\right],\\
&d_i = (s_{i+1/2} - s_{i-1/2})/(t_{i+1} - t_{i-1})\\
&s_i = \min\mathrm{mod}\left[s_{i-1/.2}, s_{i+1/2}\right]\\
&\bar{k} = 3,\ \textbf{if}\ |s_{i+1/2}| >> |s_{i-1/2}|,\ \textbf{or}\ |s_{i-1/2}| >> |s_{i+1/2}|\\
&\quad \geq 4,\ \textbf{otherwise}
\end{aligned} \quad (12)$$

At two ends, the Hyunk boundary condition will be employed [11]. As noted in Ref.[3], if a fixed factor value of $\bar{k} = 3$ is employed, this cubic interpolation might introduce too much artificial modification.

Fast Fourier Sine Transform Algorithm

In terms of the iterative filter, the FFT algorithm of Ref.[1] is modified to be the following steps:

1. Employ the iterative filter to remove the non-sinusoidal and low frequency parts.
2. For the remaining high frequency part, choose zero crossing points at two ends. Use an interpolation method to find 0 points there.
3. Use the modified monotonic cubic interpolation of Ref.[1] to regenerate the data so that total number of points are of 2^m. For a smooth data string, more than one point should be located in the range between two successive data points of the original data string to reduce interpolation error. For an oscillatory data such as the turbulent data taken by an insufficient sampling rate, more than 4 points should be considered.
4. Perform an odd function mapping with respect to one end so that the final data point is doubled.
5. A simple and fast Fourier sine transform algorithm is employed to generate the desired Fourier sine spectrum.

Since the values are chosen at two ends, the penalty of shrinking the available data range can not be avoided. Note that the odd function mapping makes the perfect periodicity of the resulting data string.

The Proposed Sharp High-Passed Filter

a. Filter Using the Iterative Gaussian Smoothing

The solution of Eqs.(10) shows that to obtain a narrow transition of the factor $A_{n,m,\sigma}$ can only be obtained via a significantly large iteration steps, say $m > 10^7$. For a practical calculation where $m \leq 150$, there will have many modes in the range of $\lambda_c < \lambda < \bar{\lambda}$ with significant amplitudes. In other words, the spectrum of the high frequency part has not a sharp cut at λ_c. One can exclude these modes to obtain a sharp high-passed filter. For the sake of completeness, the procedures are listed below.

1. Use an available FFT algorithm to estimate the Fourier spectrum of a data string.
2. Determine the cut-off frequency corresponding to λ_c and $\bar{\lambda}$.
3. Find the corresponding smooth factor σ and iteration step m by solving Eqs.(10).
4. Perform the iterative filter basing on the Gaussian smoothing method to obtain the estimated smoothed and high frequency part.
5. Use the above mentioned strategy to evaluate the Fourier sine spectrum.
6. Remove the modes whose $\lambda > \lambda_c$ and find the

resulting high frequency part. The difference between the estimated high frequency part and this high frequency part is the extracted smooth part which is equal to the inverse Fourier transform of the removed modes.

7. The summation of the estimated and extracted smooth parts is the desired smooth part.
8. The other sharp band-passed limited spectrum can be directly obtained by embedding unit block window on the spectrum. The band-passed limited data string is the corresponding inverse Fourier transform.

In order to reduce the computing time, one can choose the cutting point at λ_c with $\bar{\lambda}/\lambda_c = 4$ so that the required parameters are $(\sigma, m) \approx (0.4780, 9)$. After obtaining the high frequency part, those modes whose $\lambda \le \lambda_c$ are retained, and put modes with $\lambda > \lambda_c$ to the smoothed part.

b. Sharp Filter Using the Fourier Sine Spectrum

The previous sharp filter requires long computing time to employ the iterative Gaussian filter. The procedure of a fast and sharp filter which directly employing the Fourier sine spectrum is listed below.

1. Properly choose two end points of the data string.
2. Use the linear trend removal so that the data string has zero value at the two ends.
3. Use the above Fourier sine procedure to find the spectrum.
4. Choose the desired cut-off frequency and perform the inverse Fourier transform of the modes whose frequency is smaller than the cut-of frequency. Add this inversed data to the linear part removed by the linear trend removal. The resulting data is the smoothed part which corresponding to the smoothed part estimated by the iterative Gaussian smoothing.
5. Find the Fourier sine spectrum of the remaining high frequency part. Again, remove the modes whose frequencies are small than the cut-off frequency.

It should be noted that, it is not easy to properly choose suitable end points. Therefore, like the previous sharp filter, this filter have to discard points around the the end points. Moreover, in order to ensure an accurate estimation, the discard ranges around the two end should be larger than that of the iterative sharp filter.

Results and Discussions

The test case is shown in Fig.1 which is the u-velocity data measured at the near wake region of a low speed turbulent flow over a bluff body [12]. The measured point is at a distance of $0.5D$ from the base of the bluff body, where D is the width of the bluff body. The result of employing the original FFT algorithm is shown in Fig.2.

By choosing the cut-off frequency at 2 Hz (i.e. λ_c=0.5 sec.), and $\bar{\lambda} = 2$ sec. The solution of Eq.(9) gives $m = 10$, $\sigma = 0.3$. The resulting estimated high and low frequency parts are shown in Fig.3. After exclude the modes with $f < 2$ Hz, the resulting high frequency part is shown in Fig.4. The smoothed part extracted from those modes with $f < 2$ Hz is shown as the dotted line in Fig.5 and the final smooth part is shown as the solid line which is the summation of the extracted and estimated smooth parts. A comparison between the final spectrum shown in Fig.6 and Fig.2, it is obvious that the spectrum has sharp cut at $f = 2$ Hz. Those shown in Figs.7 are data strings in the bandwidths of $f = 2-20$ Hz, $f = 20-50$ Hz, and $f > 50$ Hz, respectively. These data strings are cut-off sharply at the cut-off frequency, say $f = 2$ Hz.

Figure 8 shows the original data string and smoothed part estimated by the Fourier sine spectrum which are corresponding to that of Fig.3. The original estimated smoothed part, smoothed part extracted from those modes with $f < 2$ Hz and final smoothed part are shown in Fig.9, which corresponding to Fig.5. Finally, as corresponding to Fig.7, data strings in the bandwidths of $f = 2-20$ Hz, $f = 20-50$ Hz, and $f > 50$ Hz, respectively, are shown in Fig.10. A carefully comparison between Figs.7 and 10 reveals that both methods give almost similar data strings of high frequency bands.

Conclusions

The simple strategy of FFT algorithm with small spectrum error and the iterative filter are successively employed to generate two sharp low-passed filters. The performances of these two sharp high/low passed filters are similar to each other.

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References

1. Y, N, Jeng, and Y. C. Cheng, "A Simple Strategy to Evaluate the Frequency Spectrum of a Time Series Data with Non-Uniform Intervals," Transactions of the Aeronautical and Astronautical Society of the Republic of China, vol.36, no.3, pp.207-214, 2004 .Bendat, J. S, and Piersol, A. G., "Random Data Analysis and Measurement Procedures," 3rd ed., John Wiley & Sons, New York, 2000, Chapters 10 & 11, pp.349-456.
2. Y. N. Jeng and Y. C. Cheng, "The New Spectrogram Evaluated by Enhanced Continuous Wavelet and Short Time Fourier Transforms via Windowing Spectrums," *Proc. 18th IPPR conference on Computer Vision, Graphics and Image Processing (CVGIP2005)*, Taipei R. O. C, Aug. 2005, pp.378-383.
3. Y. N. Jeng, C.T. Chen, and Y. C. Cheng, "A New and Effective Tool to Look into Details of a Turbulent Data String," *Proc. 12th National Computational Fluid Dyna-mics Conference*, Kaohsiung Taiwan, paper no. CFD12-2501, Aug. 2005.
4. Y. N. Jeng, C. T. Chen, and Y. C. Cheng, "Studies of Some Detailed Phenomena of a Low Speed Turbulent Flow over a Bluff Body," *Proc. 2005 AASRC/CCAS Joint Conf.*, Kaohsiung, Taiwan, Dec. 2005, paper no. H-47.
5. Y. N. Jeng, C. T. Chen, and Y. C. Cheng, "Some Detailed Information of a Low Speed Turbulent Flow over a Bluff Body Evaluated by New Time-Frequency Analysis," *AIAA* paper no.2006-3340, San Francisco June, 2006.
6. Y. N. Jeng, "Time-Frequency Plot of a Low Speed Turbulent Flow via a New Time Frequency Transformation," Proc. 16th Combustion Conf., paper no.9001, April, 2006, Taiwan.
7. Y. N. Jeng, C. T. Chen, and Y. C. Cheng, "A Time-Series Data Analyzing System Using a New Time–Frequency Transform," Proceeding of the int. conf. on innovative comput. inform. & control, Bejing, China, paper no.icicic-2006-0190, Aug. 2006.
8. Y. N. Jeng, Huang, P. G., and Chen, H., "Wave De- com position in Physical Space Using Iterative Moving Least Squares Methods," Proceedings of 11-th National Computational Fluid Dynamics Conference, Tai-Tung, paper no. CFD11-0107, Aug. 2004.
9. Y. N. Jeng[1], P. G. Huang[2], and H. Chen, "Filtering and Decomposition of Waveform in Physical Space Using Iterative Moving Least Squares Methods," AIAA paper no.2005-1303, Reno Jan. 2005.
10. Jeng, Y. N., Cheng, Y. C., and Yang, T. M., "Wave Decomposition Across Discontinuity Using Iterative Moving Least Squares Methods," Proceedings of 11-th National Computational Fluid Dynamics Conference, Tai-Tung, paper no. CFD11-0110, Aug. 2004.
11. H. T. Huynh, "Accurate Monotone Cubic Interpola- tion," SIAM J. Number. Anal. vol.30, no.1, pp57- 100, Feb.1993.
12. Wang, C. T., "Investigation of Low-Frequency Variations Embedded in Vortex Shedding Process," *Ph.D dissertation*, Department of Aeronautics and Astronautics, National Cheng-Kung University, June 2000.

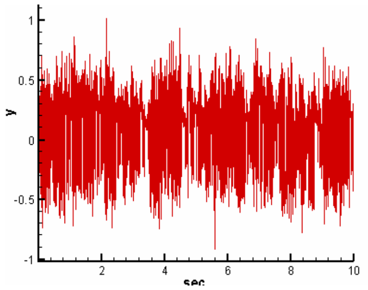


Fig.1 The original $u-$velocity data.

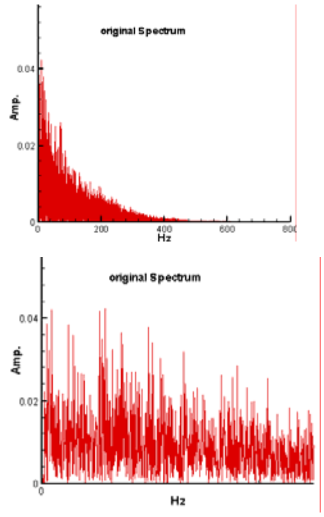


Fig.2 The original overall (top) and detailed (bottom) spectrums.

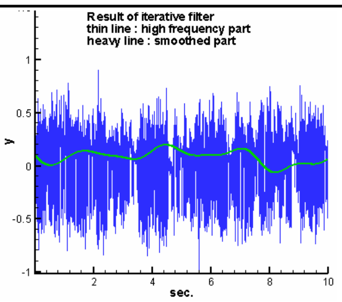


Fig.3 The result of employing the iterative filter with $m = 20$, $\sigma = 0.3$: the thin line is the high frequency part and the heavy line is the smoothed part.

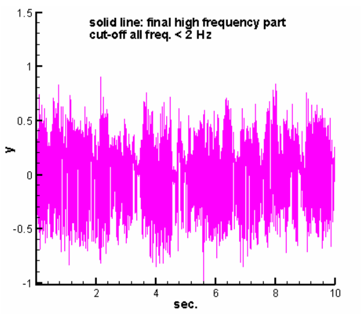


Fig.4 The final high frequency part with cut-off frequency at $f = 2$ Hz.

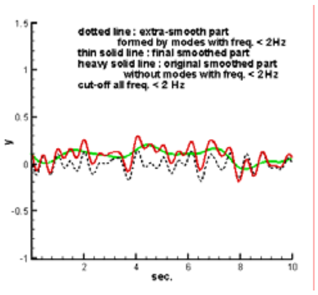


Fig.5 The smooth part generated by the first sharp filter is shown as heavy solid line; the dotted line is the extracted smooth part corresponding to those modes with $f < 2$Hz; and the final smooth part is the thin solid line.

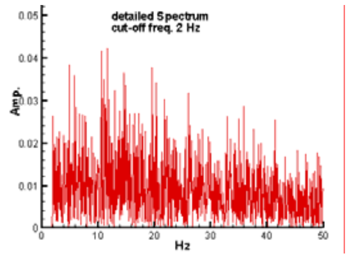
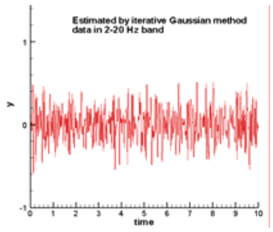
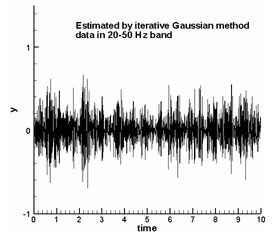


Fig.6 The detailed plot of the final spectrum.

(a)



(b)



(c)

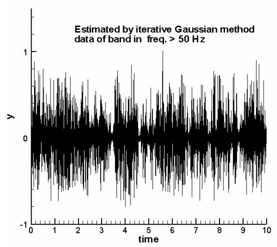


Fig.7 The data strings with finite band-widths, find by the first sharp filter: (a) 2-20 Hz, (b) 20-50Hz, and (c) frequency larger than 50Hz.

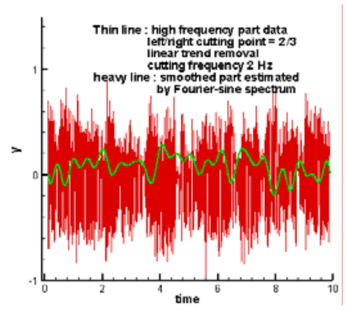


Fig.8 The first result of employing the second iterative filter with linear trend removal and inverse data from low frequency modes of the Fourier sine spectrum: the thin line is the high frequency part and the heavy line is the estimated smoothed part.

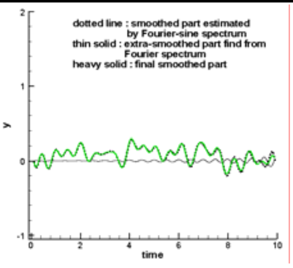


Fig.9 The smooth part generated by the second sharp filter is shown as heavy dotted line; the thin solid line is the extracted smooth part corresponding to those modes with $f < 2$Hz; and the final smooth part is the heavy solid line.

(a)

(b)

(c)

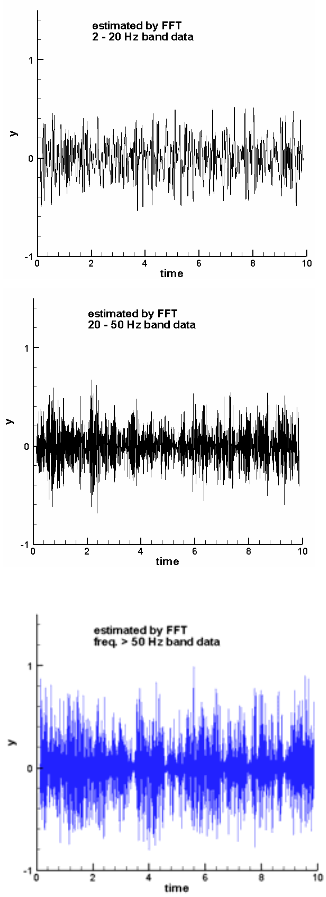


Fig.10 The data strings with finite band-widths, find by the second sharp filter: (a) 2-20 Hz, (b) 20-50Hz, and (c) frequency larger than 50Hz.

時變數據的尖銳型高低通濾波器

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摘要

本文使用疊代型濾波器配合簡易的傅式正弦函數轉換式，發展兩個尖瑞的高低通濾波器。第一種濾波器使用疊代型濾波器是一種略加修改以得到非負值的擴散特性，經決定切除頻率後，疊代型濾波器提供近似的高低頻數據。應用傅式正弦函數轉換式得到頻率後，再切除長於切除頻率之波後，便可得到尖銳的低通濾波結果。第二種濾波器先用線性內插法使兩端數據成為0,求取 Fourier sine 頻譜後,將低頻部份加回線性內插部份而成非週期加上低頻的數據,以取代疊代型濾波器之低頻數據，兩種濾波器都可得到尖銳切除的高低通頻帶，且速度還算合理。

關鍵詞：尖銳濾波高低通器，疊代型濾波器，傳式正弦函數轉換。