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育能 教授鈞鑒：

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表 14

A Short Time Fourier Transform For A Time Series Data String

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ABSTRACT

An existing Fast Fourier Transform (FFT) algorithm with small spectrum error is modified to construct the two-dimensional plot of the Short Time Fourier Transforms (SFTF). Before the data string is treated by the FFT algorithm, the iterative filter via the Gaussian smoothing is applied to remove the undesired non-sinusoidal part and wave components whose frequencies are not in the interested band. The proposed short time Fourier spectrum is evaluated in a rectangular window whose end points are defined by zero crossing positions of the high frequency data. After sweeping all the designed windows and designating the corresponding spectrum to the center of the window, a two-dimensional time-frequency data set is constructed. This two-dimensional plot is an approach more direct than the Gabor transform and is somewhat more complicated than a simple short time Fourier expansion. As comparing with these transforms, the proposed transform does not modify the data as the former one and every window has zero ends as comparing with the latter. A test case examines the beat wave formed by 3 known sine waves and shows that the zero crossing point can not be put at region where the approximate amplitude of the beat attains local minimum value. Two additional test cases are employed to show the capability of capturing local properties of this new time-frequency analysis. It seems that, because of the uncertainty of the time-frequency analysis via the Fourier series expansion and the non-uniform window width, the resulting time-spectrum plot evaluated via a relative short window width arrangement shows a faded character in spite of the capability of capturing temporary property. For a sufficiently large window size, the plot shows a steady and clear quality but loses the capability of capturing temporary property.

Keywords: FFT with small error, time frequency analysis.

1. INTRODUCTION

Because of the rapid development of computer hardware and software, the capability of collecting huge number of long data string increases rapidly. Generally, a real data string frequently has a complex structure which changes main characters rapidly. It is well known that the classical Fourier spectrum can not explicitly reflect the time dependent information. In order to look into the local property, in addition to the simple STFT, the wavelet and Gabor transforms are widely applied tools [1,2]. Although both wavelet and Gabor transforms have being extensively studied from mathematical point of view, these methods are approximate expansions rather than an exact mapping procedure such as the Fourier spectrum. On the other hand, the simple STFT is a direct Fourier transform but is suffered from the error introduced by the non-periodicity of the local window. Therefore, it is a worthwhile effort to develop a short time Fourier transformation with small error.

A valuable application of a STFT with small error is to do data repairing. Today, there are many sensors put at outdoor to collect environmental data such as underground water level, sea level, atmospheric temperature, and atmospheric pressure, etc. Another example is the two- and three-dimensional particle image velocimetries. One cannot guarantee that there is not data missing period. Generally, the main characters of these data can be doubly checked by other sources. If all the related data show that a data should be in normal condition during its data missing period, it is reasonable to assume that all variations of amplitude, frequency, and phase angle of different modes are all continuous. In such a situation, a STFT with small error is an effective tool to repair the data with an acceptable error.

For a data string, the spectrum evaluated by a Fourier expansion may involve significant low frequency error because of the existence of the non-sinusoidal part and non-periodicity at two ends. The error induced by the non-sinusoidal part is frequently named as the Direct Current (DC) contamination. In a previous study [3], a simple procedure had been proposed to remedy error induced by these factors.

In two following studies [5,6], the iterative filter (basing on the Gaussian smoothing and cubic moving least squares method) without phase error was proposed. If wave components of a data string have an obvious gap on spectrum domain, the filter can effectively give the low and high frequency parts at the gap’s two sides. This result shows that the cubic moving least squares method employed in Ref.[3] can not effectively remove all the undesired low frequency wave components from the high frequency part. This study will employ the iterative filter of Ref.[5,6] to remove the non-sinusoidal and low frequency parts. Moreover, because of the FFT algorithm gives a spectrum with small spectrum error, it will be employed to obtain a new STFT method.

2. ANALYSIS

2.1 Classical and Short Time Fourier Transformations

Assume that a discrete data string $y_j = y(j\Delta t), j = 1,2,..., N-1$ can be approximated by

$$y(t) \approx \sum_{n=0}^{M-1} b_n \cos\left(\frac{2\pi t}{\lambda_n}\right) + c_n \sin\left(\frac{2\pi t}{\lambda_n}\right) \qquad (1),$$

where $M \le N$, and λ_n’s may be wavelengths of different wave components. This equation becomes the well known discrete Fourier transform [1,2] if $\lambda_n = T/n, M = N$, and $0 \le t < T$, where T is the expansion interval and n is the mode index.

In fact, the discrete Fourier expansion enforces the assumption of $y(T) = y(0)$, $y'(T) = y'(0)$, $y''(T) = y''(0)$, ..., $y^{(M-1)}(0) = y^{(M-1)}(T)$, where the superscript of the last equality denotes the $(M-1)$-th order of derivative. Once the periodicity of a typical periodic function $y(t)$ does not occur at $t = 0$ and $t = T$, the Fourier expansion evaluated at the interval of $0 \le t < T$ is certainly different from the original periodic $y(t)$ because of $\lambda_n \ne T/n$. For a discrete and periodic data string, even if the periodicity precisely occurs at $t = 0$ and $t = T$, the chance to satisfy $\lambda_n = T$ exactly is very small because of limited number of data point. Since a data string may be formed from multiple sources and may contain non-sinusoidal part, the resulting spectrums $\{a_n, b_n\}$, $n = 0,1,..., N-1$, may involve the following information: non-sinusoidal and sinusoidal parts of original data and the artificially extra-terms at $t = T$, say, $y(0) - y(T)$, $y'(0) - y'(T)$, … etc. The following content will demonstrate how to employ the iterative filter basing on the Gaussian smoothing method and strategy of Ref.[3] to eliminate these undesired drawbacks.

As noted in Ref.[7], the Fourier spectrum is a parameter representation of the original data string in an averaged sense. It cannot reflect the temporary variation of spectrum which is interested in many respects. The simplest method to reflect the time dependent spectrum is the simple STFT [1,2]. This transform defines a moving rectangular window, say $t \in [t_k, t_{k+L}]$ where $L < N-1$, and find the spectrum for successive k's. The resulting spectrum is denoted as the spectrum at central points, $p = k + L/2$. Therefore, the resulting spectrum is obtained in the range of $p = L/2, 1 + L/2, 2 + L/2, ..., N-1-L/2$. In practical application several modifications upon window arrangements had been shown in Ref.[2]. This approximation has three properties: (1) a_n and b_n of Eq.(1) now becomes $a_n(t_p)$ and $b_n(t_p)$; and (2) since $M = L < N-1$, the mode number is reduced accordingly; and (3) λ_n's are defined in terms of $L \cdot \Delta t/n$. Since the periodic condition of the corresponding data segment is hard to satisfy, this STFT has certain spectrum error. For the speech signal processing, such error is not very important for speech perception and can be suppressed to certain extent [2].

The Gabor transform [1], which employs a Gaussian window or else to eliminate the importance of the periodic condition, is a widely applied extension. The drawback of non-periodic condition of the previous STFT is effectively suppressed by the embedded window. For example, at a point where $t = t_L = L\Delta t$, $a_n(t_L)$ and $b_n(t_L)$ are evaluated from the original data weighted by the Gaussian function, say $y(t_k)\exp[-(k-L)^2(\Delta t)^2/(2\sigma^2)]$, rather than the original string $y(t_k)$. The resulting spectrum is interpreted as the spectrum of the central point t_L of the moving window.

Since the available data points are finite, the resulting transform for points with $0 \le L \le \sqrt{2}\sigma$, and $N-1 \ge L \ge N-1-\sqrt{2}\sigma$ may have relatively large error. Because the window runs over the whole range, the total mode number is not changed, say $M = N$.

Both the simple STFT and Gabor transforms give temporary spectrums rather than fixed spectrum with respect to time, say $a_n(t)$ and $b_n(t)$. Therefore, these resulting spectrums are also named as time-frequency transforms. Generally speaking, if the window size is larger the resolution of frequency will be better, and the resulting time-frequency transform is named as a narrowband spectrograms. On the other hand, a shorter window size generates a worse frequency resolution, the resulting transform is called a wideband spectrogram.

Similar transforms are the discrete and continuous wavelet transforms [1,2]. Unlike the above mentioned short time Fourier transforms with a fixed window size for different frequencies, the window size of a wavelet transform is closely related to wavelength (and hence frequency) of the spectrum. Nevertheless, their performances are not much different from each other. All of these time-frequency transforms become more and more popular because they do reflect time dependent information.

2.2 The Proposed Short Time Fourier Transform

The result of Ref.[3] shows that to properly choice a periodic data and artificially make it completely periodic is a effective method to remove the DC contamination of non-sinusoidal part. The section will show the improvement over the strategy of Ref.[3] and apply it to local segments. Like the STFT and Gabor transforms, the resulting local spectrum is defined as the spectrum at the central point of a segment. The detailed algorithm will be introduced in the following content. For the sake of completeness, all the related topics are briefly listed below.

2.2.1 The Iterative Filter Basing on Gaussian Smoothing

In Ref.[5,6], it was proven that after applying the Gaussian smoothing once, the resulting smoothed data becomes

$$\bar{y}_1(t) \approx \sum_{n=0}^{N-1} a(\sigma/\lambda_n)\{b_n \cos\left(\frac{2\pi t}{\lambda_n}\right) + c_n \sin\left(\frac{2\pi t}{\lambda_n}\right)\} \tag{2}$$

where $a(\sigma/\lambda_n)$'s are the attenuation factors introduced by the smoothing. If the data is periodic, it can be proven numerically that

$$0 \le a(\sigma/\lambda_n) = \sum_{k=-\infty}^{\infty} \cos\frac{2\pi t_k}{\lambda_n} e^{-t_k{}^2/(2\sigma^2)} \approx \exp[-2\pi^2\sigma^2/\lambda_n^2] \le 1 \tag{3}$$

For convenient, the removed high frequency part is denoted as $y_1^{'}$. After applying the same smoothing to $y_1^{'}$, the new smoothed result and remaining high frequency part are obtained and are denoted as $\bar{y}_2$ and $y_2^{'}$, respectively, where $y_2^{'} = y_1^{'} - \bar{y}_2$. Then, repeat the same procedure will obtain the $m-$th smoothed and high frequency parts, say $\bar{y}_m$ and $y_m^{'}$, respectively. The following relation can be built via simple algebraic manipulation.

$$y_m^{'} = \sum_{n=0}^{N}[1-a(\sigma/\lambda_n)]^m\left[b_n\cos\left(\frac{2\pi t}{\lambda_n}\right)+c_n\sin\left(\frac{2\pi t}{\lambda_n}\right)\right]$$

$$\begin{aligned}\bar{y}(m) &= \bar{y}_1+\bar{y}_2+...+\bar{y}_m = y - y_m^{'}\\ &= \sum_{n=0}^{N}A_{n,m,\sigma}\left[b_n\cos\left(\frac{2\pi t}{\lambda_n}\right)+c_n\sin\left(\frac{2\pi t}{\lambda_n}\right)\right]\end{aligned} \tag{4}$$

$\bar{y}(m)$ can be considered as the smoothed part and $y_m^{'}$ as the high frequency part. After substituting Eq.(3), the attenuation factor of employing the Gaussian smoothing method satisfies

$$\begin{aligned}&A_{n,m,\sigma} \approx 1-[1-\exp\{-2\pi^2\sigma^2/\lambda_n{}^2)]^m\\ &0 \le A_{n,m,\sigma} \le 1\end{aligned} \tag{5}$$

Obviously, the accumulated smooth part is embedded by a diffusive attenuation factor without phase error. It was also proven in Ref.[5,6] that the transition region from $A_{n,m,\sigma}$ = 0 to 1 is much narrower than that of the original $a(\sigma/\lambda_n)$ for a sufficiently large iteration step m.

Suppose that all the waveforms within the range of $\lambda_{c1} < \lambda < \lambda_{c2}$ are insignificantly small. The above mentioned iterative smoothing procedure can be an effective filter to give both the low and high frequency parts. The desired parameters σ and number of iteration steps m are solved by the following simultaneous equations.

$$\begin{aligned}&1-[1-\exp(-2\pi^2\sigma^2/\lambda_{c1}{}^2)]^m = B_1\\ &1-[1-\exp(-2\pi^2\sigma^2/\lambda_{c2}{}^2)]^m = B_2\end{aligned} \tag{6}$$

where $B_1, B_2 = 0.001, 0.999$, respectively, are employed in this study.

The solution of these simultaneous solution shows that the both σ and m depends on $\lambda_{c2}/\lambda_{c1}$ and m quickly approaches to a huge number as $\lambda_{c2}/\lambda_{c1} \to 1$. For example, with $B_1, B_2 = 0.001, 0.999$, $\lambda_{c2}/\lambda_{c1}$=3, $m \approx 18$ and $\sigma = 0.7018155$; $\lambda_{c2}/\lambda_{c1}$=2, $m \approx 127$ and $\sigma \approx 0.77155$; $\lambda_{c2}/\lambda_{c1}$=1.5, $m \approx 8135$ and $\sigma \approx 0.897181$; and $\lambda_{c2}/\lambda_{c1}$=1.25, $m \approx 46{,}179{,}655$ and $\sigma \approx 1.1153$. Therefore, it seems that to take the ratio $\lambda_{c2}/\lambda_{c1} \ge 2$ is a practical approach.

Before the FFT algorithm of Ref.[3] or the above mentioned FFT algorithm is employed. One can use a classical FFT algorithm to obtain a spectrum. After inspecting both the original data and spectrum, he can decide a suitable transition range defined by λ_{c1} and λ_{c2} to remove the non-sinusoidal and low frequency part. Finally, the expansion of original data becomes

$$y(t) = \bar{y}(t) + y_H(t) \approx \bar{y}(t) + \sum_{n>c1}^{M-1} b_n \cos\left(\frac{2\pi t}{\lambda_n}\right) + c_n \sin\left(\frac{2\pi t}{\lambda_n}\right) \tag{7}$$

where $\bar{y}(t)$ is the smoothed part and $y_H(t)$ is the high frequency part to be represented either by Fourier spectrum or by the proposed STFT. Since the physical meaning of $\bar{y}(t)$ can be easily identified, this decomposition is more appropriate than that of Eq.(1). If a very sharp filter is desired, the data associated to modes between λ_{c1} and λ_{c2} can be added to $\bar{y}(t)$ and the subscript of the summation index becomes $n \geq c2$ to exclude these modes. In the following content, the spectrums of $y_H(t)$ is employed to construct the proposed STFT expansion.

Monotonic Cubic Interpolation Method

In Ref.[4], a slope limited Hermite cubic interpolation between points (x_i, x_{i+1}) is employed to construct several essentially non-oscillatory cubic interpolations. Consider the Hermite cubic interpolation, say

$$\begin{aligned}
&y(t) = c_3(t-t_i)^3 + c_2(t-t_i)^2 + c_1(t-t_i) + c_0 \\
&c_0 = y_i,\ c_1 = y_i^{'},\ s_{i+1/2} = (y_{i+1} - y_i)/(t_{i+1} - t_i) \\
&c_2 = \frac{3s_{i+1/2} - 2y_i^{'} - y_{i+1}^{'}}{t_{i+1} - t_i},\ c_3 = \frac{y_i^{'} + y_{i+1}^{'} - 2s_{i+1/2}}{(t_{i+1} - t_i)^2}
\end{aligned} \tag{8}$$

Ref.[3] modified the M3A interpolation of Ref.[4] to give necessary limiter to discrete slopes $y_i^{'}, y_{i+1}^{'}$, say,

$$\begin{aligned}
&y_i^{'} = \operatorname{sgn}(\tilde{t}_i)\min\left[\frac{1}{2}\left|p_{i-1/2}^{'}(t_i) + p_{i+1/2}^{'}(t_i)\right|, \max\left\{k\,|s_i|, \frac{k}{2}|\tilde{t}_i|\right\}\right] \\
&p_{i-1/2}^{'}(t_i) = s_{i-1/2} + d_{i-1/2}(t_i - t_{i-1}) \\
&p_{i+1/2}^{'}(t_i) = s_{i+1/2} + d_{i-1/2}(t_i - t_{i+1}) \\
&\tilde{t}_i = \min\bmod\, p_{i-1/2}^{'}(t_i), p_{i+1/2}^{'}(t_i)] \\
&d_{i+1/2} = \min\bmod[d_i, d_{i+1}] \\
&d_i = (s_{i+1/2} - s_{i-1/2})/(t_{i+1} - t_{i-1}),\ s_i = \min\bmod[s_{i-1/2}, s_{i+1/2}] \\
&k = 3,\ \text{if } |s_{i+1/2}| >> |s_{i-1/2}|, \text{or } |s_{i-1/2}| >> |s_{i+1/2}| \\
&\quad \geq 4, \text{ otherwise}
\end{aligned} \tag{9}$$

At two ends, the Hyunk boundary condition will be employed [4]. This algorithm is employed to redistribute a non-uniformly distributed data into equally spaced data whose data points are exactly equal to 2^m.

A Fast Fourier Transform Algorithm with small error

In terms of the iterative filter, the FFT algorithm of Ref.[3] is modified to be the following steps:

1. Employ the iterative filter to remove the non-sinusoidal and low frequency parts.
2. In the remaining high frequency part, choose zero crossing points next to two ends. Use an interpolation method to find 0 points there.
3. Use the above mentioned monotonic cubic interpolation to regenerate the data so that total number of points are of 2^m. For a smooth data string, more than one point should be located in the range between two successive data points of the original data string to reduce interpolation error. For an oscillatory data such as the turbulent data taken by an insufficient sampling rate, more than 2 points should be considered.
4. Perform an odd function mapping with respect to one end so that the final data point is doubled.
5. A simple and fast Fourier transform algorithm is employed to generate the desired spectrum.

Since zero values are chosen at two ends, the penalty of shrinking the available data range can not be avoided. Note that the odd function mapping leads to a perfect periodicity of the resulting data string.

A Proposed Short Time Fourier Transform

If the high frequency part $y_H(t)$ of Eq.(7) is directly employed to construct the STFT, zero-crossing points associated to the high frequency part may be improperly addressed. A careful inspection upon data string combining different modes reveals that nonzero low frequency modes may significantly reduce the number of zero-crossing point and nonzero high frequency modes may shift zero-crossing location. Therefore, a band passed data string is suitable for a given frequency. For example, for modes with wavelengths λ's, the desired band is $\lambda_1 < \lambda < \lambda_2$. According to the authors' experience, both the amplitude and frequency variation of a mode with wavelength λ is reflected by the variation in regions of (λ_1, λ) and (λ, λ_2) at two sides of λ, respectively. In order to compromise the requirements of zero-crossing points, one should employ a bandwidth with moderate distances of $\lambda - \lambda_1$ and $\lambda_2 - \lambda$.

With the above mentioned FFT algorithm as a tool, this study employs the following strategy to construct the short time Fourier transforms.

1. Use the iterative filter to remove the smooth and low frequency part.
2. For a given frequency, find the corresponding band-limited data string either by applying the iterative filter or by imposing a proper window upon the spectrum of $y_H(t)$ plus an inverse FFT.

3. For the band-limited data string, identify all the zero crossing points, say t_i^0's, in terms of a searching procedure together with a suitable interpolation method.
4. After assign a proper window size ΔS, starting from the left end, define a series of successive windows T_i whose left end points are t_i^0. The right end point of window T_i is chosen at t_{i+k}^0, $k > 1$ and satisfies the criterion that $\left\| t_{i+k}^0 - t_i^0 \right| - \Delta S \right|$ is minimum with respect to all positive integer k.
5. Perform the above FFT algorithm to evaluate the spectrum for each window T_i. Let the resulting spectrum be designated to the central point $\bar{t}_k = (t_i^0 + t_{i+k}^0)/2$. The spectrum is interpreted by the following frequency.

$$\hat{f}_{n,k} = n/[2(t_{i+k}^0 - t_i^0)] \tag{10}$$

where n is the mode number index and the factor 2 in the denominator results from the odd function mapping procedure.

6. In order to obtain a simple graphical display, the resulting spectrum at different $\bar{t}_k$ and $\hat{f}_{n,k}$ are linearly interpolated onto a uniform grid system on (t_i, $f_{n,k}$) plane.

Since the window size $t_{i+k}^0 - t_i^0$ differs from each other, the resulting time frequency spectrum shows an oscillatory behavior obviously whenever the window size is not wide enough as will be discussed later.

The Uncertainty Properties of Fourier Spectrum [1,2]

It is well known that all the time frequency transform have the limitation of uncertainty principle, such that the product of effective signal duration σ_t^2 and bandwidth σ_ω^2 cannot be less than 1/4, where σ_t^2 and σ_ω^2 are variances of time and frequency resolutions, respectively. It means that for a small expansion window size (corresponding to a good temporal resolution of local property), the accuracy of frequency is will be bad. On the other hand, a large window width with worse temporal resolution leads to an accurate frequency resolution. Consider the Gabor transform as an example, if a large window size is employed, the resulting Gabor coefficient plot is called the narrowband spectrogram which is used for good frequency resolution. On the other hand, the wideband spectrogram uses a small window size which allows good temporal resolution of signals. The present STFT is belong to the class of time frequency transform so that it posses the similar limitation.

3. RESULTS AND DISCUSSIONS

In Ref.[3], there is not example with known function and exact spectrum to demonstrate the merit of the proposed FFT algorithm. For the sake of completeness, the following function is employed to explicitly examine the resulting error of the proposed FFT algorithm.

$$\begin{aligned} y(x) &= \bar{y}(x) + y_1(x) \\ \bar{y}(x) &= 1 + 2\bar{x} + \bar{x}^2/2 + 0.3\exp[-0.05x^2]\sin(6\pi\bar{x}) \\ y_1(x) &= 0.3\exp[-0.5\bar{x}^2](1 + 2\bar{x} + \bar{x}^2)\sin(32\pi\bar{x}) \\ &\quad + 0.2\sin(56\pi\bar{x}) + 0.4\sin(100\pi\bar{x}) \end{aligned} \tag{11}$$

where $\bar{x} = x/10$, $y(x)$ is the composite waveform (shown as the thin solid line in Fig.1), $\bar{y}(x)$ involves a non-sinusoidal and low frequency part (dotted line) and $y_1(x)$ is composed of two simple short waves and a complicated short wave (shown as thin solid line around $x = 0$). The data is expressed in a uniform spacing in the range of $0 \le x \le 10$ so that the total number of point is 8192+1. The first 8192 points are employed to find the spectrum via a simple FFT program. After apply the iterative filter, the estimated smooth part is shown as long dashed line, and the high frequency part is the dashed line. Obviously, the estimated smooth part deviates from the given smooth part and their difference becomes significant around two ends. Figure.2 shows three spectrums: the thin solid line with diamond symbol is the spectrum of $y(x)$, the heavy solid line with delta symbol is the spectrum of $y_1(x)$ plus an odd function mapping with respect to the point at $x = 10$ so that the total width is doubled. The line with gradient symbol is the spectrum of the estimated high frequency part via the iterative filter. The low frequency error of $y(x)$'s spectrum is induced by both the non-sinusoidal part of $\bar{y}(x)$ and the non-periodic condition at two ends. By using the modified FFT algorithm to treat the high frequency data $y_1(x)$, the resulting spectrum does not involve the non-sinusoidal part and the error induced by $y(10) \neq y(0)$. The exact spectrum is achieved because $y_1(0) = y(20)$ and all the available conditions of derivatives at two ends, say $y_1'(0) = y_1'(20)$, $y_1''(0) = y_1''(20), \ldots$ etc. are satisfied. It is seen that the estimated spectrum captures all the dominate modes. The small disagreement comes from the deviation of estimated smooth part from the original smooth part around two ends. This test case shows that the present FFT algorithm does significantly reduce the spectrum error as compare with the result of classical FFT algorithm.

According to authors' experience of many test studies, if the composite wave forms a beat, the resulting Fourier spectrum converges slowly for an improper choice of end points. In order to demonstrate such property, the following composite wave is examined as a first test case.

$$\begin{aligned} y(x) &= \sin\frac{46\pi x}{24} + 0.9\sin\frac{50\pi x}{24} + 0.1\sin(2\pi x) \\ 0 &\le x \le 365 \text{ days} \end{aligned} \tag{12}$$

The detailed plot the original data is shown in Fig.3 which is in form of a beat wave. Figures 4 show two pictures of local short time Fourier spectrum whose approximate segment intervals are 30 and 225 days, respectively. From these two figures, it is clear that the local short time Fourier series evaluated by end points located at successive zero crossing points involves certain degree of error. For convenient, the line connecting local maximum points are considered as the upper limiting envelope and that formed by the local minimum points is designed as the lower limiting envelope. The approximate amplitude of the beat, r, is then defined as the half distance between lower and upper limiting envelopes. It is obvious that approximate amplitude of Fig.3 shows a period of about 16 days. A careful inspection upon Figs.3 and 4 reveals that the data around $r \approx$ minimum is not of regular shape and the spectrums evaluated by a window with end points around these regions shown a local splitting property. For a short expansion period of 30 days, the spectrum of points with $r \approx$ minimum has a serious error and even has a multiple solution. The result of Fig.4b which uses the expansion range of 225 days, the multiple solution character still presents. In other words, a window with end point around minimum radius will introduce a large spectrum error. Suppose the end points only take that around $r \approx$ maximum, the resulting spectrums with periods of 32 and 96 days are shown in Figs.5a and 5b, respectively. This test case clearly shows that, one has better to choose end points at zero crossing points where the composite wave has a regular shape and smooth upper and lower envelopes. Any small wavy behavior of envelopes around a zero crossing point will lead to spectrum error.

The second test case is the tide wave data of Hobihu habor at Pin-Dong (in south part of Taiwan). The data range runs from January 1/2001 to December 31/2001. Figure.6a shows the original data (around sea level = 1000 mm) and the whole day tide data (around sea level = 0 mm). The latter wave is obtained by employing the iterative filter with $m = 30$ iterations and $\sigma = 4$ days to remove the low frequency part and $m = 200$ iterations and $\sigma = 0.35$ days to exclude the high frequency part. The resulting whole day tide wave data shows a beat wave of period 15 days which is about half of a lunar month as shown in the detailed plot of Fig.6b. This is a practical data with a complicated composite beat wave and is a band-passed data filtered from the original data. The resulting spectrum is shown in Fig.7.

In Fig.6, the data around regions of minimum radius is not a regular variation and is even more complicated than that shown in Fig3. According to the conclusion of the previous test case, it is better to exclude those spectrum estimated by the segments whose end points located around these regions. With the restriction of this rule, a series of two-dimensional amplitude plots of wave width vs. time are shown in Fig.8 with different window sizes. Because of many waves combining together, the window sizes (determined by zero-crossing points) are not uniform. These spectrums show that the increase of expansion range does decrease the spectrum error and the limit is the spectrum shown in Fig.7. A first glance upon Fig.6b shows that the upper and lower limiting envelopes of the beat wave is not smooth. Therefore, the resulting dominate modes are not in form of a perfectly shaped impulses. In other words, resulting bands of dominate modes show a faded character.

The third test case is the voice data of “hello” of a 10 years old child. The data and spectrum are shown in Fig.9a and 9b, respectively. In this case the band passed filter is obtained by two stages: $\sigma = 1$ second, $m = 1$ iteration to remove low frequency part and $\sigma = 0.001$ seconds, $m = 200$ iterations to remove the high frequency part. In Fig.10a-d, the approximate window widths are 0.016, 0.02, 0.031, and 0.061 seconds, respectively. For a speech signal, Fig.10 can be considered as a series of spectrograms with different window sizes that are corresponding to different window widths. A careful inspection upon all the figures of Fig.10 reveals that the spectrogram with window width 0.02 second (Fig.9b) gives a best resolution. The spectrogram with window width 0.016 second (Fig.10a) gives an excellent resolution of rapid frequency changing part but the band width of the constant frequency part is bigger than that of Fig.10b. On the other hand, those use window widths of 0.031 and 0.062 seconds obtain a good resolution at frequency constant part but show a serious dilution of band resolution in rapid frequency changing part. In other words, this test case shows that the present short time Fourier transforms follows the uncertainty principle of Ref.[1,2].

The last two examples show the following properties of the present STFT, say

1. If a beat wave has a complicated shape of upper and lower envelope, the resulting time varying spectrum shows a scattered behavior.
2. For a data string composed of complicated wave components whose frequencies change rapidly, there is not a suitable window size.

Nevertheless, the proposed STFT does faithfully reflect the temporary variation of spectrum.

4. CONCLUSIONS

The simple strategy of FFT algorithm with small spectrum error and the iterative filter is employed to generate a true short time Fourier transforms. A test case of given sine waves shows that one should not choose an end point at zero crossing points around which approximate radius of the beat attains minimum value. A tide wave data is employed the capability of resolving temporal behavior of a composite wave with a short widow width. The other test case shows that the present transform is restricted by the uncertainty principle of most time-frequency analysis tools. It is believed that this short time Fourier transforms is a convenient tool to reflect the local spectrum of a time series data with complex structure. A direct application of this STFT to repair a data string with short data missing periods is on the way.

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Fig.1 The test function: original composite wave is thin solid line, given smooth part is the dotted line; given high frequency part is thin solid line around $x(t) = 0$; the long dashed line is the smooth part estimated by the iterative filter with $\sigma = 1, m = 30$; and the dashed line around $x(t) = 0$ is the estimated high frequency part.

Fig.2 The resulting spectrums: diamond symbol is the spectrum evaluated from $y(x)$; delta symbol is the exact spectrum; and the gradient symbol is the spectrum estimated by the high frequency part of Fig.1.

Fig.3 The detailed plot of the test function of Eq.(12).

Fig.4 The resulting spectrums of Fig.3: (a) $\mathrm{T_{total}} \approx 30$ days and (b) $\mathrm{T_{total}} \approx 225$ days.

Fig.5 The resulting spectrums: (a) $\mathrm{T_{total}} \approx 32$ days and (b) $\mathrm{T_{total}} \approx 96$ days with two end points located at $r \approx r_{\max}$.

Fig.6 The tide wave data of Hobihu habor (Pingtung, Taiwan): (a) the upper line around 1000 mm is the original data; the line around 0mm is the whole day tide data (filtered by σ = 4, 30 iterations and $\sigma = 0.35$, 200 iterations) and (b) the detailed plot of the whole day tide data.

Fig. 7 The spectrum of the whole day tide data of Fig.6 of Fig.6.

Fig.8The two-dimensional amplitude plots of short time Fourier transform with different interval sizes, evaluated by the band-passed data of Fig.6: (a) width ≈ 30 days; (b) 45 days; (c) 60 days; (d) 75 days; (e) 90 days; and (f) 120 days.

Fig.9 The data and spectrum of three voice “hello”: (a) the raw data of voice “hello”; and (b) the spectrum.

Fig.10 The two-dimensional amplitude plots of low frequency part of the voice “hello”: (a) interval length is about 0.016 seconds; (b) 0.02 seconds; (c) 0.031 seconds; and (d) 0.061 seconds. The band passed data is generated by steps: $\sigma = 1, m = 1$ to remove low frequency part and $\sigma = 0.001$ seconds, $m = 200$ iterations to remove the high frequency part.